# atmcs10

ALGEBRAIC TOPOLOGY METHODS, COMPUTATION & SCIENCE

# WELCOME

It is a great pleasure to host you for the 10th event in the conference series on Algebraic Topology: Methods, Computation, and Science (ATMCS10). We follow the strong tradition in this series, that started over twenty years ago in Stanford, of bringing together leading established researchers and young scientists in this emerging discipline, providing an opportunity for the exchange of knowledge and the development of new ideas. After the ATMCS09 had to be moved on line two years ago, we feel fortunate to be able to hold this meeting in person, and bring the community together after a long time.

There are many elements that have to come together for a successful conference. We are grateful to all our speakers and poster presenters, and to our Scientific Committee that selected them after careful consideration. It is an exciting programme and we are looking forward to the talks and posters. The conference is sponsored by the Centre for Topological Data Analysis, and you will find among the participants many Oxford members of the Centre in conference T-shirts ready to help you; we thank all our young helpers and colleagues for their support. We also like to thank Matt Kahle for his lead on the NSF application through which many of our participants from the US are supported. Our special personal thanks go to Nicola Kirkham who many of you will have corresponded with; she has been the bedrock of the conference office. Last but not least we thank you all for coming to Oxford.

We wish you a stimulating and productive week here at the Mathematical Institute in Oxford.

Prof. Ulrike Tillmann FRS and Prof. Heather Harrington
Directors of the Centre of Topological Data Analysis, and local organisers

**WIFI**

If you do not have access to Eduroam, any visitor to the building just needs to select the 'The Cloud' wifi SSID. If you have used this on your device in other places (e.g. stations, pubs, event venues etc) then you will already have registered; if you are new to it then open a web browser and it takes you to a registration page after which you are connected.

# BREAKFAST

Served 8.30 - 9.30am in AWB Mezzanine

**MONDAY**

Alden's Butchers' cured bacon bap
Alden's Butchers' sausage bap
Roast field mushroom ciabatta (vg)
Coffee, tea and herbal
Selection of juices

**TUESDAY**

Butter croissant (v)
Mini Danish pastries, butter, and fruit preserves (v)
Coconut yoghurt with fruit compote (vg)
Coffee, tea and herbal
Selection of juices

**WEDNESDAY**

A selection of butter croissants and bagels (v)
Coconut yoghurt with seasonal fruit compote (vg)
Smoked salmon, smoked ham and a selection of British cheese
Freshly baked breads (v)
Butter, and fruit preserves
Coffee, tea and herbal
Selection of juices

**THURSDAY**

Breakfast Boards to share 3-4 people
Coconut yoghurt, large croissants, sliced fruit, fresh orange juice
Honey mustard glazed ham, mature cheddar, butter croissants, overnight oats, selection of sliced fruits, artisan demi-baguettee

# LUNCH

Recommendations of local places to eat

**BRANCA**
*111 Walton St, Oxford OX2 6AJ*
Vibrant Italian eatery with exposed bricks and a terrace serving small plates and stone-baked pizza.

**JERICHO CAFE**
*112 Walton St, Oxford OX2 6AJ*
Your friendly neighbourhood family-run cafe

**VAULTS & GARDEN CAFE**
*1 Radcliffe Sq University Church, Oxford OX1 4AJ*
Simple organic food served in a quintessential Oxford setting.

**PICNIC IN THE PARK**
**University Parks**
S Parks Rd, Oxford OX1 3RF

**Wellington Square Gardens**
Oxford, OX1 2JD

**Port Meadow**
Off Walton Well Road, OX2 6ED

**TooGoodToGo - Free app**
From supermarkets to sushi, nearby stores that have unsold, surplus food up for grabs. Rescue surprise bags filled with delicious food sold at 1/3 price.

# DINNER

We have reserved a few tables for you at our favourite restaurants in the area. Just pick where you'd like to go, then sign up at the registration desk.

**MONDAY**

**JAMAL'S**
107 - 108 Walton St, OX2 6AJ
Quality Indian curries

**MAMA MIA JERICHO**
102 Walton St, Oxford OX2 6EB
Italian Restaurant & Pizzeria

**PIERRE VICTORIE**
9 Little Clarendon St, OX1 2HP
Classic French bistro

**RICKETY PRESS**
67 Cranham St, OX2 6DE
Pub Food & great pizza

**ZHENG**
82 Walton St, Oxford OX2 6EA
South-east Asian fusion food

**WEDNESDAY**

**THE VICTORIA**
90 Walton St, Oxford OX2 6EB
Classic period tavern with food and garden

**THE GARDENERS ARMS**
39 Plantation Rd, Oxford OX2 6JE
Buzzy pub with a garden and veggie food

**GIGGLING SQUID**
55 Walton St, Oxford OX2 6AE
Thai restaurant

**THE WHITE RABBIT**
21 Friars Entry, Oxford OX1 2BY
Great beers & Amazing pizza




# EXCURSIONS

Sign up at the registration desk



## BLENHEIM PALACE

9.15am - 2.15pm

Meet at AWB to take the coach to Blenheim Palace. Take a walk in Capability Brown's gardens, and enjoy the annual flower festival. There is a cafe and restaurant in the palace, or you could take a picnic.

## STONEHENGE & THE WHITE HORSE

8.15am - 5pm

Meet at AWB to take the coach to the prehistoric monument on Salisbury Plain in Wiltshire, and on the way back visit the Bronze Age white horse and iron age hill fort in Uffington.
The Stonehenge audio tour is now available to download for free. Please click here.

## WALKING TOUR OF OXFORD

9am (90 mins) 15 people max
10am (90mins) 15 peope max

Starting and ending at the Weston Library. On this walking tour you will see the interior of the Divinity School, the exterior of the Bodleian and Radcliffe Camera, and the streets surrounding the central Bodleian site. The tour guides are excellent.

## OXFORD PUNT

10am-12pm

Meet at the Cherwell Boathouse for a 2 hour punt along the River Cherwell taking a view of Oxford colleges from the river.

# 20.6

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|---|---|
| **9.30-10.30** | **HERBERT EDELSBRUNNER**<br>Depth in arrangements: Dehn--Sommerville--Euler relations with applications |
| | **BREAK** |
| **11.00-12.00** | **ULRICH BAUER**<br>Persistent homology for functionals |
| | **GROUP PHOTO/LUNCH** |
| **2.00-3.00** | **ERIC SEDGEWICK**<br>How to draw a knot |
| | **BREAK** |
| **3.30-4.30** | **KATHARINE TURNER**<br>Theory and applications of the Persistent Homology Transform and variants - an overview |
| **4.45** | **POSTER SESSION & DRINKS RECEPTION** |

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|---|---|---|---|
| **9.30 - 10.30** | **ANDREW BLUMBERG**<br>Probabilistic stability theorems for multiparameter persistent homology | | |
| | **BREAK** | | |
| **11.00 - 12.30** | **MICHAEL ADAMER**<br>The magnitude vector of images | **BENEDIKT FLUHR**<br>Categorification of Extended Persistence Diagrams | **SHREYA ARYA**<br>A sheaf-theoretic construction of shape space |
| | **BARBARA GIUNTI**<br>Average complexity of persistence algorithms for clique filtrations | **VADIM LEBOVICI**<br>Hybrid transforms of constructible functions with applications to multiparameter persistent magnitude | **ALEXANDER WAGNER**<br>Distributed Persistence: Inverse Theorems and Dimensionality Reduction |
| | **IRIS YOON**<br>Persistent Extension and Analogous Bars: Data-Induced Relations Between Persistence Barcodes | **HENRY ADAMS**<br>The Persistent Topology of Optimal Transport Based Metric Thickenings | **GREGORY HENSELMAN-PETRUSEK**<br>Beyond field coefficients: saecular barcodes and generators for persistent homology |
| | **LUNCH** | | |
| **2.00 - 3.00** | **SAUGATA BASU**<br>Complexity of computing homology of semi-algebraic sets and mappings | | |
| | **BREAK** | | |
| **3.30 - 4.30** | **ELIZABETH MUNCH**<br>The Many Faces of the Interleaving Distance | | |
| | **BREAK** | | |
| **4.45 - 5.45** | **GUNNAR CARLSSON**<br>TDA and motion planning | | |
| | **7.00 PM CONFERENCE DINNER, BALLIOL COLLEGE** | | |

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|---|---|---|---|
| **9.30-10.30** | **HÉLÈNE BARCELO**<br>Discrete cubical homotopy groups and real K(π, 1) spaces | | |
| | **BREAK** | | |
| **11.00-12.30** | **INGRID MEMBRILLO-SOLIS**<br>Tracking the time evolution of soft matter systems via structural heterogeneity | **MELVIN VAUPEL**<br>Section complexes of simplicial height functions | **OMER BOBROWSKI**<br>Universal Distribtuion of Persistent Cycles |
| | **ANDREAS OTT**<br>A persistent homology approach for the surveillance of emerging adaptive mutations in the evolution of the coronavirus | **BIANCA DORNELAS**<br>Sparse Higher Order Cech Filtrations | **AMBROSE YIM**<br>Local Inference of Morse Indices from Finite Point Samples |
| | **PARKER EDWARDS**<br>Quantifying topological features in microscopy images | **ANNA SCHENFISCH**<br>Ordering Topological Descriptor Types | **TADAS TEMČINAS**<br>Multivariate Normal Approximations for Simplex Counts in Random Complexes |
| | **LUNCH** | | |
| **2.00 -3.00** | **FACUNDO MEMOLI**<br>The Gromov-Hausdorff distance between spheres | | |
| | **BREAK** | | |
| **3.30 - 4.30** | **PETER BUBENIK**<br>Persistent homology using filtered closure spaces | | |
| | **7.00PM CONFERENCE DINNER RESERVATIONS - SIGN UP AT THE REGISTRATION DESK** | | |

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|---|---|---|---|
| **9.30-10.30** | **LEONID POLTEROVICH**<br>Applying topological data analysis to pure mathematics | | |
| | **BREAK** | | |
| **11.00-12.30** | **ANNA-LAURA SATTELBERGER**<br>The Shift-Dimension: an Algebraic Invariant of Multipersistence Modules | **MARZIEH EIDI**<br>Topological Learning from Dynamics on Data | **PAUL DUNCAN**<br>Homological percolation on a torus |
| | **SAMANTHA MOORE**<br>The Generalized Persistence Diagram Encodes the Bigraded Betti Numbers | **WOOJIN KIM**<br>Extracting Persistent Clusters in Dynamic Data via Möbius inversion | **ERIKA ROLDAN**<br>Topology of random 2-dimensional cubical complexes |
| | **ADAM ONUS**<br>Quantifying the Homology of Periodic Simplicial Complexes | **KELLY MAGGS**<br>Signal Compression and Reconstruction on Chain Complexes with Morsified Deformation Retracts | **FRANCISCO MARTINEZ-FIGUEROA**<br>The chromatic number of random Borsuk graphs |
| | **LUNCH** | | |
| **2.00 -3.00** | **NINA OTTER**<br>What are weather regimes? A topologist's answer | | |
| | **BREAK** | | |
| **3.30 - 4.30** | **VIN DE SILVA**<br>Thoughts on Teaching Topology | | |
| | **4.30PM A COLLECTIVE STROLL TO THE VICTORIA ARMS PUB ON THE BANKS OF THE RIVER CHERWELL FOR DRINKS & DINNER. THE MILL LANE, OLD MARSTON, OX3 0QA** | | |

# POSTERS

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|---|---|---|---|
| 1. | HASSAN ABDALLAH | STATISTICAL INFERENCE FOR PERSISTENT HOMOLOGY APPLIED TO SIMULATED FMRI TIME SERIES DATA | WAYNE STATE UNIVERSITY |
| 2. | ÁNGEL JAVIER ALONSO | REDUCING MULTI-PARAMETER FLAG FILTRATIONS VIA EDGE COLLAPSES | TU GRAZ |
| 3. | HÅVARD BAKKE BJERKEVIK | $\ELL^P$-CONTINUITY PROPERTIES OF MULTICOVER PERSISTENT HOMOLOGY | TU GRAZ |
| 4. | MATÍAS BENDER | COMPUTING MINIMAL PRESENTATIONS OF MULTI-PARAMETER PERSISTENT HOMOLOGY | TECHNISCHE UNIVERSITÄT BERLIN |
| 5. | NICOLAS BERKOUK | PROJECTED BARCODES : A NEW CLASS OF INVARIANTS AND DISTANCES FOR MULTIPARAMETER PERSISTENCE MODULES | EPFL |
| 6. | ADAM BROWN | LEARNING HOMOLOGICAL STRATIFICATIONS FROM FINITE SAMPLES | ISTA |
| 7. | JOHNATHAN BUSH | VIETORIS-RIPS COMPLEXES, PROJECTIVE CODES, AND ZEROS OF ODD MAPS | UNIVERSITY OF FLORIDA |
| 8. | YUEQI CAO | A CONDITION FOR THE UNIQUENESS OF FRÉCHET MEANS OF PERSISTENCE DIAGRAMS | IMPERIAL COLLEGE LONDON |
| 9. | MAURICIO CHE | METRIC GEOMETRY OF SPACES OF PERSISTENCE DIAGRAMS | DURHAM UNIVERSITY |
| 10. | DOMINIC DES JARDINS CÔTÉ | FROM FINITE VECTOR FIELD DATA TO FORMAN'S COMBINATORIAL DYNAMICAL SYSTEMS | UNIVERSITY OF SHERBROOKE |
| 11. | LIU ENHAO | CURSE OF DIMENSIONALITY IN PERSISTENCE DIAGRAMS | KYOTO UNIVERSITY |
| 12. | XIMENA FERNANDEZ | TOPOLOGY OF THE NEURAL ACTIVITY OF GRID CELLS | DURHAM UNIVERSITY |
| 13. | CHRISTOPHER FILLMORE | A CAUTIONARY TALE: BURNING THE MEDIAL AXIS IS UNSTABLE | IST AUSTRIA |
| 14. | ANA LUCIA GARCIA PULIDO | ON THE GEOMETRY OF THE SPACE OF PERSISTENCE BARCODES | UNIVERSITY OF LIVERPOOL |
| 15. | ADÉLIE GARIN | FROM TREES TO BARCODES AND BACK AGAIN: COMBINATORIAL AND PROBABILISTIC PERSPECTIVES | EPFL / AALBORG UNIVERSITY |
| 16. | MARIO GOMEZ FLORES | CURVATURE SETS OVER PERSISTENCE DIAGRAMS | THE OHIO STATE UNIVERSITY |
| 17. | ANDREA GUIDOLIN | STABLE HOMOLOGICAL INVARIANTS FROM WASSERSTEIN METRICS | KTH ROYAL INSTITUTE OF TECHNOLOGY |
| 18. | ABIGAIL HICKOK | DENSITY-SCALED FILTERED COMPLEXES | UCLA |
| 19. | RENEE HOEKZEMA | MULTISCALE SPECTRAL METHODS FOR FEATURE SELECTION IN SINGLE CELL DATA | UNIVERSITY OF OXFORD |
| 20. | EMILE JACQUARD | THE SPACE OF BARCODE BASES OF A PERSISTENCE MODULE | UNIVERSITY OF OXFORD |
| 21. | VITALIY KURLIN | PERSISTENCE VS NEWER ISOMETRY INVARIANTS OF POINT SETS | UNIVERSITY OF LIVERPOOL |
| 22. | DARRICK LEE | A TOPOLOGICAL APPROACH TO MAPPING SPACE SIGNATURES | EPFL |

# POSTERS

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| 23. | SUNHYUK LIM | VIETORIS-RIPS PERSISTENT HOMOLOGY, INJECTIVE METRIC SPACES, AND THE FILLING RADIUS | MAX PLANCK INSTITUTE |
| 24. | FRANK H. LUTZ | RANDOM SIMPLE-HOMOTOPY THEORY | TU BERLIN |
| 25. | KILLIAN MEEHAN | TOPOLOGICALLY LEARNED EMBEDDINGS AND APPLICATIONS TO CHROMOSOME STRUCTURAL ANALYSIS | KYOTO UNIVERSITY |
| 26. | ADAM ONUS | QUANTIFYING THE HOMOLOGY OF PERIODIC SIMPLICIAL COMPLEXES | QUEEN MARY |
| 27. | EDUARDO PALUZO-HIDALGO | SIMPLICIAL-MAP NEURAL NETWORKS | UNIVERSIDAD DE SEVILLA |
| 28. | SARAH PERCIVAL | USING MAPPER TO REVEAL MORPHOLOGICAL RELATIONSHIPS IN PASSIFLORA LEAVES | MICHIGAN STATE UNIVERSITY |
| 29. | ABHISHEK RATHOD | EFFICIENT APPROXIMATION OF THE MULTICOVER BIFILTRATION | PURDUE UNIVERSITY |
| 30. | YOHAI REANI | PERSISTENT CYCLE REGISTRATION AND TOPOLOGICAL BOOTSTRAP | TECHNION |
| 31. | RAPHAEL REINAUER | PERSFORMER: A TRANSFORMER ARCHITECTURE FOR TOPOLOGICAL MACHINE LEARNING | EPFL |
| 32. | SARA SCARAMUCCIA | BARCODE MAPPINGS INDUCED BY PERSISTENCE MODULE MORPHISMS | UNIVERSITÀ DEGLI STUDI DI ROMA |
| 33. | NICHOLAS SCOVILLE | THE HOMOTOPY TYPE OF THE MORSE COMPLEX FOR SOME COLLECTIONS OF TREES | URSINUS COLLEGE |
| 34. | CHUNYIN SIU | DETECTION OF SMALL TOPOLOGICAL FEATURES BY THE SCALE-INVARIANT ROBUST DENSITY-AWARE DISTANCE (RDAD) FILTRATION | CORNELL UNIVERSITY |
| 35. | ANNA SONG | SIGNED DISTANCE PERSISTENT HOMOLOGY OF TUBULAR AND MEMBRANOUS SHAPES | IMPERIAL COLLEGE LONDON |
| 36. | MANUEL SORIANO-TRIGUEROS | INDUCING A PARTIAL MATCHING FROM A MAP BETWEEN PERSISTENCE MODULES | UNIVESIDAD DE SEVILLA |
| 37. | DANIEL SPITZ | THE GEOMETRY OF NONEQUILIBRIUM SELF-SIMILAR BEHAVIOR IN GAUGE THEORIES | UNIVERSITY OF HEIDELBERG |
| 38. | RAPHAËL TINARRAGE | SIMPLICIAL APPROXIMATION TO CW COMPLEXES IN PRACTICE | FGV EMAP |
| 39. | FRANCESCA TOMBARI | REALISATIONS OF POSETS AND TAMENESS | KTH |
| 40. | LUKAS WAAS | PERSISTENT STRATIFIED HOMOTOPY TYPES | HEIDELBERG UNIVERSITY |
| 41. | QIQUAN WANG | FUNCTIONAL SUPPORT VECTOR MACHINE ON PERSISTENT HOMOLOGY RANK FUNCTIONS | IMPERIAL COLLEGE LONDON |
| 42. | MATHIJS WINTRAECKEN | A SHORT PROOF OF THE HOMOTOPY RECONSTRUCTION RESULT BY NIYOGI, SMALE AND WEINBERGER FOR SETS OF POSITIVE REACH WITH TIGHT BOUNDS | IST AUSTRIA |
| 43. | CHENGUANG XU | COMPUTING INTERVAL DECOMPOSITIONS/APPROXIMATIONS FOR COMMUTATIVE LADDER PERSISTENT HOMOLOGY | KYOTO UNIVERSITY |
| 44. | JUN YOSHIDA | SIMPLICIAL COMPLEXES IN TOPOSES AND APPLICATION TO TIME-DEPENDENT PERSISTENT HOMOLOGY | RIKEN AIP |
| 45. | LING ZHOU | PERSISTENT HOMOTOPY GROUPS OF METRIC SPACES | THE OHIO STATE UNIVERSITY |
| 46. | BARBARA GIUNTI | AMPLITUDES IN MULTIPARAMETER PERSISTENCE | TU GRAZ |
| 47. | ARAS ASAAD | PERSISTENT HOMOLOGY FOR BREAST TUMOR CLASSIFICATION USING MAMMOGRAM SCANS | UNIVERSITY OF BUCKINGHAM |

# Day 1
# 20.6

# Depth in arrangements: Dehn–Sommerville–Euler relations with applications.

Herbert Edelsbrunner

**Abstract**

The depth of a cell in an arrangement of $n$ (non-vertical) great-spheres in $S^d$ is the number of great-spheres that pass above the cell. We prove Euler-type relations, which imply extensions of the classic Dehn–Sommerville relations for convex polytopes to sublevel sets of the depth function, and we use the relations to extend the expressions for the number of faces of neighborly polytopes to the number of cells of levels in neighborly arrangements.

This is work with Ranita Biswas, Sebastiano Cultrera, and Morteza Saghafian.

# Persistent homology for functionals.

Ulrich Bauer

**Abstract**

I will illustrate the central role and the historical development of persistent homology beyond applied topology, connecting recent developments in persistence theory with classical results in critical point theory and the calculus of variations. Presenting recent joint work with M. Schmahl and A. Medina-Mardones, I will explain how the modern view on persistence provides a new and clarifying perspective on Morse's theory of functional topology, which has been instrumental in the first proof of the existence of unstable minimal surfaces by Morse and Tompkins.

# How to draw a knot.

Eric Sedgwick

**Abstract**

Knots are traditionally presented via planar diagrams. Haken's solution to the unknotting problem shows the computational advantage of another representation, a triangulation of the complement of the knot. We explore the relationship between these two representations and how the number of crossings in a diagram relates to the number of tetrahedra in a triangulation. While it is straightforward to convert a knot diagram to a triangulation, we demonstrate a solution to the reverse problem: Given a triangulation of a knot complement, how do you draw the knot? The solution relies on normal surface theory, especially on ideas from the Rubinstein-Thompson algorithm for 3-sphere recognition.

This is joint work with Robert Haraway, Neil Hoffman and Saul Schleimer.

# Theory and applications of the Persistent Homology Transform and variants - an overview

Katharine Turner

**Abstract**

Persistent homology and other topological summaries like Euler curves are a classic way to characterise geometric shape. One common filtration is a height function which will capture information about geometry of a shape with respect to a specific direction. The Persistent Homology Transform (PHT) basically expands on this idea by considering the height functions in all directions. This has nice theoretical properties, in particular that it completely describes a compact nice subsets of Euclidean space, and also can provide new metrics for quantitatively measuring the difference between geometric objects. We can also make different variants of the PHT by using different topological summaries, most notably the Euler Characteristic Transform which constructs the Euler Curves in each direction. This talk will be a bit of a survey of both theory developments and also interesting applications.

# Day 2
# 21.6

# Probabilistic stability theorems for multiparameter persistent homology

Andrew Blumberg

**Abstract**

The depth of a cell in an arrangement of n (non-vertical) great-spheres in $S^d$ is the number of great-spheres that pass above the cell. We prove Euler-type relations, which imply extensions of the classic Dehn–Sommerville relations for convex polytopes to sublevel sets of the depth function, and we use the relations to extend the expressions for the number of faces of neighborly polytopes to the number of cells of levels in neighborly arrangements.

This is work with Ranita Biswas, Sebastiano Cultrera, and Morteza Saghafian.

# The magnitude vector of images

Michael F. Adamer[1, 2], Leslie O'Bray[1, 2], Edward De Brouwer[3], Bastian Rieck[1, 2, 4, †], Karsten Borgwardt[1, 2, †]

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[2]SIB Swiss Institute of Bioinformatics, Switzerland

[3]ESAT-STADIUS, KU LEUVEN, 3001 Leuven, Belgium

[4]B.R. is now with the Institute of AI for Health, Helmholtz Zentrum München, Neuherberg, Germany

[†]These authors jointly supervised this work.

**Abstract**

The magnitude of a finite metric space is a recently-introduced invariant quantity [Leinster, 2010], which encodes many invariants from integral geometry and geometric measure theory. As shown by Leinster [2010], the magnitude of a metric space can intuitively be understood as an attempt to measure its *effective size*.

Despite extensive theoretical work on magnitude, and the close connections between magnitude and persistent homology [Otter, 2018], many of its properties are still unknown, and numerous open questions remain concerning more practical implications, in particular in topological data analysis and machine learning. Recent work [Bunch et al., 2021] has made a step towards linking magnitude vectors with machine learning applications, however, this branch of research is still in its infancy.

We instead investigate the properties of magnitude vectors on individual images, with each image forming its own finite metric space. We therefore consider the magnitude vectors of each individual data point by endowing each image with a metric space structure and explore its properties. In this work, we derive a novel algorithm that significantly speeds up the computation of magnitude on images and empirically show its correctness. This computational relaxation paves the way for magnitude to be used more broadly in machine learning research. We further incorporate a magnitude computation into a differentiable neural network layer that can be easily integrated into existing deep learning architectures.

The applicability of magnitude in various domains is investigated, with a particular emphasis on edge detection, both to directly perform edge detection in images, as well as serving as a pre-processing step that highlights edges in an image before being fed to an edge-detection algorithm.

Finally, we investigate the properties of the magnitude *function* of images by stacking different magnitude vectors and its possibilities of improving classification performance of existing computer vision methods.

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## References


Tom Leinster. The magnitude of metric spaces. *arXiv preprint arXiv:1012.5857*, 2010.

Nina Otter. Magnitude meets persistence. homology theories for filtered simplicial sets. *arXiv preprint arXiv:1807.01540*, 2018.

Eric Bunch, Jeffery Kline, Daniel Dickinson, Suhaas Bhat, and Glenn Fung. Weighting vectors for machine learning: numerical harmonic analysis applied to boundary detection. *arXiv preprint arXiv:2106.00827*, 2021.




# Categorification of Extended Persistence Diagrams


Ulrich Bauer

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Benedikt Fluhr

Department of Mathematics, Technical University of Munich (TUM), Germany, fluhr@ma.tum.de


January 31, 2022


The *extended persistence diagram* introduced by Cohen-Steiner, Edelsbrunner, and Harer [CEH09] is an invariant of real-valued continuous functions, which are $\mathbb{F}$-*tame* in the sense that all open interlevel sets have degree-wise finite-dimensional cohomology with coefficients in a fixed field $\mathbb{F}$. We categorify this invariant in the sense that we provide a functor $h$ from the category of $\mathbb{F}$-tame functions to an abelian Frobenius category $\mathcal{A}$ with the following property. For an $\mathbb{F}$-tame function $f\colon X \to \mathbb{R}$ the extended persistence diagram $\mathrm{Dgm}(f)$ uniquely determines - and is determined by - the corresponding element $[h(f)] \in K_0(\mathcal{A})$ in the Grothendieck group $K_0(\mathcal{A})$ of the abelian category $\mathcal{A}$. This is in close analogy to the following categorification of the Euler characteristic: Given a topological space $X$ the Euler characteristic $\chi(X) \in \mathbb{Z}$ uniquely determines $[\Delta_\bullet(X)] \in K_0(\mathrm{Ab})$, which is the element in the Grothendieck group $K_0(\mathrm{Ab})$ of the category of abelian groups Ab corresponding to the singular chain complex $\Delta_\bullet(X)$ of $X$. We hope this provides a new perspective to persistence diagrams in settings where structural results are unavailable.

Our construction of this categorification builds on our results from [BBF21]. More specifically, $h(f) \in \mathrm{Ob}(\mathcal{A})$ is an invariant we introduced there as the *relative interlevel set cohomology (RISC)* of $f\colon X \to \mathbb{R}$. As an intermediate step we harness our structure result from [BBF21] to show that $\mathcal{A}$ is the Abelianization of the category of derived sheaves on $\mathbb{R}$, which are *tame* in the sense that sheaf cohomology of any open interval is finite-dimensional in each degree. This yields a close link between derived level set persistence [Cur14; KS18] and the categorification of extended persistence diagrams.

## References

[BBF21] Ulrich Bauer, Magnus Bakke Botnan, and Benedikt Fluhr. "Structure and Interleavings of Relative Interlevel Set Cohomology". Dec. 2021. arXiv: 2108.09298 [math.AT].

[CEH09] David Cohen-Steiner, Herbert Edelsbrunner, and John Harer. "Extending persistence using Poincaré and Lefschetz duality". In: *Found. Comput. Math.* 9.1 (2009), pp. 79–103. ISSN: 1615-3375. DOI: 10.1007/s10208-008-9027-z. URL: https://doi.org/10.1007/s10208-008-9027-z.

[Cur14] Justin Michael Curry. *Sheaves, cosheaves and applications*. Thesis (Ph.D.)–University of Pennsylvania. ProQuest LLC, Ann Arbor, MI, 2014, p. 317. ISBN: 978-1303-96615-6. URL: http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqm&rft_dat=xri:pqdiss:3623819.

[KS18] Masaki Kashiwara and Pierre Schapira. "Persistent homology and microlocal sheaf theory". In: *J. Appl. Comput. Topol.* 2.1-2 (2018), pp. 83–113. ISSN: 2367-1726. DOI: 10.1007/s41468-018-0019-z. URL: https://doi.org/10.1007/s41468-018-0019-z.




# A SHEAF-THEORETIC CONSTRUCTION OF SHAPE SPACE

SHREYA ARYA, JUSTIN CURRY, AND SAYAN MUKHERJEE

Abstract. We present a sheaf-theoretic construction of shape space—the space of all shapes. We do this by describing an $\infty$-sheaf on the poset category of o-minimal sets, where objects of this category are mapped to their Persistent Homology Transforms (PHTs), which are viewed as derived sheaves on $\mathbb{S}^{d-1} \times \mathbb{R}$. A recent result [16, 10] that builds on work of Schapira, showed that this transform is injective, thus making the PHT a good summary object for each shape. By passing to sheaves adapted to $\infty$-categories, we are able to "glue" PHTs of different shapes together to build up the PHT of a larger shape. This is necessary because the ordinary derived category of sheaves is not abelian and lacks limits [18]. Once this obstacle is circumvented, we prove a new approximation result that degree-0 information is enough, i.e. PHT$^0$, is sufficient to compute PHT$^n$ for any $n$ up to some prescribed tolerance.

## 1. Administrative Details

Shreya Arya is a graduate student at Duke and would be the designated speaker if selected to talk at ATMCS 10.


## References

[1] Michael Artin, Grothendieck Topologies. Notes on a Seminar by M. Artin. Spring, 1962, Harvard University, Department of Mathematics.
[2] P. Bendich, J.S. Marron, E. Miller, A. Pieloch, S. Skwerer. Persistent homology analysis of brain artery trees. Annals of Applied Statistics. 2016 03;10(1):198–218.
[3] Bredon, Glen E. Sheaf theory. Vol. 170. Springer Science & Business Media, 2012.
[4] F.L. Bookstein. Morphometric Tools for Landmark Data: Geometry and Biology. Cambridge University Press, 1997.
[5] D.M. Boyer, Y. Lipman, E. St. Clair, J. Puente, B.A. Patel, T. Funkhouser, J. Jernvall, and I. Daubechies. Algorithms to automatically quantify the geometric similarity of anatomical surfaces. Proceedings of the National Academy of Sciences U.S.A. 2011;108(45):18221.
[6] D.M. Boyer,J. Puente, J.T. Gladman, C. Glynn,S. Mukherjee, G.S. Yapuncich, I. Daubechies. A new fully automated approach for aligning and comparing shapes. The Anatomical Record. 2015; 298(1):249–276.
[7] D.M. Boyer, G.F. Gunnell, S. Kaufman, T.M. McGeary. Morphosource: archiving and sharing 3-D digital specimen data. The Paleontological Society Papers. 2016;22:157–181.
[8] L. Crawford, A. Monod A, A.X. Chen, S. Mukherjee S, A. Rabadán. Predicting Clinical Outcomes in Glioblastoma: An Application of Topological and Functional Data Analysis. Journal of the American Statistical Association. 2020;115-531:1139-1150.
[9] Curry, J., Hang, H., Mio, W., Needham, T., & Okutan, O. B. (2021). Decorated Merge Trees for Persistent Topology. arXiv preprint arXiv:2103.15804.
[10] J. Curry, S. Mukherjee, K. Turner. How many directions determine a shape and other sufficiency results for two topological transforms. https://arxiv.org/pdf/1805.09782.
[11] Paul Dupuis and Ulf Grenander. Variational problems on flows of diffeomorphisms for image matching. Quarterly of Applied Mathematics, LVI(3):587–600, September 1998.
[12] T. Gao, S.Z. Kovalsky, I. Daubechies. Gaussian process landmarking on manifolds. SIAM J Math Data Sci. 2019;1(1):208–236.

[13] A. Goswami. Phenome10K: a free online repository for 3-D scans of biological and palaeontological specimens; 2015.
[14] Edelsbrunner, H. (1993, July). The union of balls and its dual shape. In Proceedings of the ninth annual symposium on Computational geometry (pp. 218-231).
[15] Topologie algébrique et théorie des faisceaux, Roger Godement, Publications de, 1 (1958)
[16] R. Ghrist, R. Levanger, H. Mai H. Persistent homology and Euler integral transforms. Journal of Applied and Computational Topology. 2018;2(1-2):55–60.
[17] Hatcher, Allen. Algebraic Topology. Cambridge University Press, 2001
[18] Hörmann, Fritz. The ∞-categorical interpretation of Abelian and non-Abelian derived functors. https://www.uni-due.de/ hx0050/pdf/infty.pdf
[19] Cohomology of sheaves, Iversen, Birger, 2012, Springer Science & Business Media
[20] Kashiwara, Masaki, and Pierre Schapira. Sheaves on Manifolds: With a Short History. Les débuts de la théorie des faisceaux. By Christian Houzel. Vol. 292. Springer Science & Business Media, 2013.
[21] D.G. Kendall. The diffusion of shape. Advances in Applied Probability. 9(3):428–430, 1977.
[22] D.G. Kendall. Shape Manifolds, Procrustean Metrics, and Complex Projective Spaces. Bulletin of the London Mathematical Society, 16(2):81–121, 1984.
[23] Miller E. Fruit flies and moduli: interactions between biology and mathematics. Notices of the AMS. 2015;62(10):1178–1184.
[24] Niyogi, P., Smale, S., & Weinberger, S. (2008). Finding the homology of submanifolds with high confidence from random samples. Discrete & Computational Geometry, 39(1-3), 419-441.
[25] M. Ovsjanikov, M. Ben-Chen, J. Solomon, A. Butscher, and L. Guibas. Functional maps: a flexible representation of maps between shapes. ACM Trans Graph. 2012;31(4):30:1–30:11.
[26] W.S. Tang, G. Monteiro da Silva, H. Kirveslahti, E. Skeens, B. Feng, T. Sudijono, K.K. Yang, S. Mukherjee, B. Rubenstein, L. Crawford. 2020;bioRxiv 2021.07.28.454240;
[27] K. Turner, S. Mukherjee, D. Boyer. Persistent Homology Transform for Modelling Shapes and Surfaces. Information and Inference: A Journal of the IMA. 2014;3(4):310-344.
[28] L. Van den Dries, Tame Topology and O-Minimal Structures, Cambridge University Press, 1998.
[29] B. Wang, T. Sudijono, H. Kirveslahti, T. Gao, D.M. Boyer, S. Mukherjee, L. Crawford. A statistical pipeline for identifying physical features that differentiate classes of 3D shapes. The Annals of Applied Statistics. 2021;15(2):638–661.
[30] Lurie, Jacob. Higher algebra. (2012): 80
[31] Dugger, Daniel, Sharon Hollander, and Daniel C. Isaksen. Hypercovers and simplicial presheaves. Mathematical Proceedings of the Cambridge Philosophical Society. Vol. 136. No. 1. Cambridge University Press, 2004.
[32] Jacob Lurie, Higher Topos Theory, Annals of Mathematics Studies 170, Princeton University Press 2009.

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# Average complexity of persistence algorithms for clique filtrations

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Since persistent homology has proven to be a powerful tool in data analysis, the algorithmsfor its computation are essential. The majority of these algorithms are (efficient) variants of the left-to-right matrix column reduction, and they have worst-case complexity cubical in the number of input simplices. Even if carefully constructed filtrations can achieve the worst-case complexity, empirical evidence suggests a much faster average behaviour. In this talk, we present the first theoretical study of the algorithmic complexity of computing the persistent homology of a randomly chosen filtration. Specifically, we prove upper bounds for the average fill-up (number of non-zero entries) of the boundary matrix on Erdös-Renyi filtrations and Vietoris-Rips filtrations after matrix reduction. Our bounds show that, in both cases, the reduced matrix is expected to be significantly sparser than what the general worst-case predicts. Our method is based on previous results on the expected first Betti numbers of corresponding complexes. We establish a link between these results and the boundary matrix's fill-up. This link holds for all clique filtrations. Thus, our bounds can be expanded to other degrees and other filtrations once suitable results on the corresponding expected Betti numbers are provided. Moreover, we show using some benchmarks that, for Vietoris-Rips complexes, our bound is asymptotically tight up to logarithmic factors. Finally, we construct an Erdös-Renyi filtration achieving the worst-case fill-up and complexity.

# Hybrid transforms of constructible functions with applications to multiparameter persistent magnitude

Abstract submission for ATMCS 10

Vadim Lebovici*

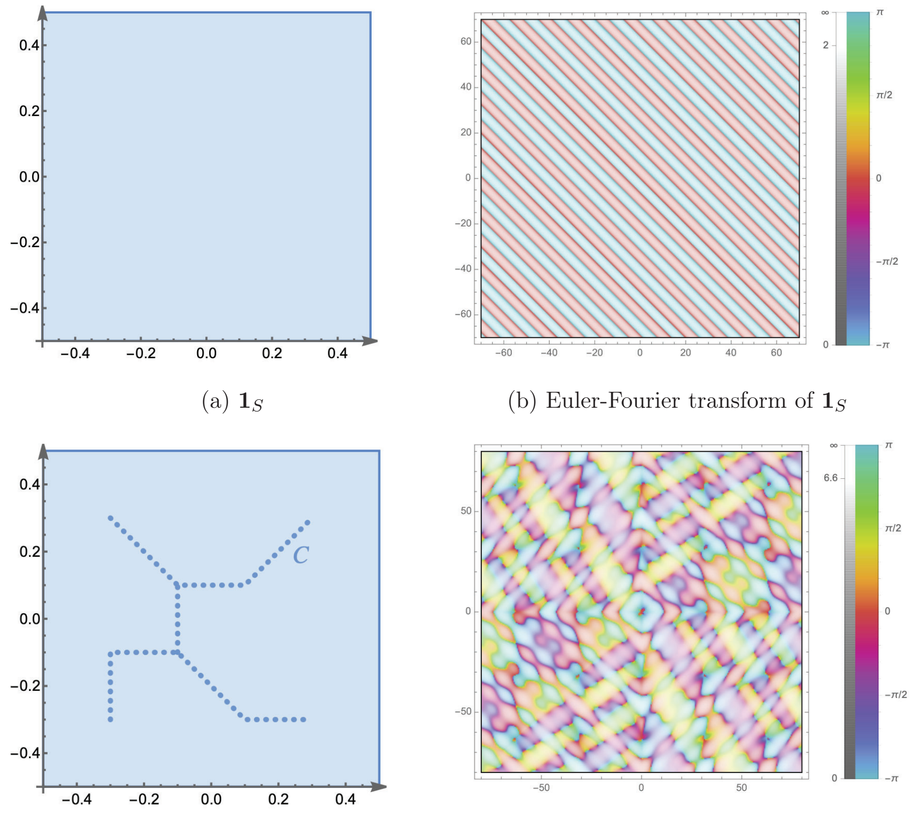
January 18, 2022

**Abstract.** Euler calculus techniques — integration of constructible functions with respect to the Euler characteristic — have led to important advances in topological data analysis. For instance, the (constructible) Radon transform has provided a positive answer to the following question: are two subsets of $\mathbb{R}^n$ with same persistent homology in all degrees and for all height filtrations equal? More generally, the constructible functions naturally associated to multi-parameter persistent modules stand as simple, informative and well-behaved, albeit incomplete, invariants of these objects.

Following my recent work [Leb21], I will introduce integral transforms combining Lebesgue integration and Euler calculus for constructible functions and present two main outcomes. The first is a generalization of Govc and Hepworth's *persistent magnitude* to multi-parameter persistent modules. The second is a mean formula for such transforms in the context of sublevel-sets persistent homology for random filtrations. More generally, I will expose how Lebesgue integration gives access to well-studied kernels and to regularity results, while Euler calculus conveys topological information and allows for compatibility with operations on constructible functions (convolution, pushforward, etc). Focusing on two examples, the Euler-Fourier and Euler-Laplace transforms, I will show various examples illustrating that they are strictly more discriminating than their classical analogues. See Figure 1 for an illustration.[1]

## References

[Leb21] Vadim Lebovici. *Hybrid transforms of constructible functions*. 2021. arXiv preprint: arXiv:2111.07829.




(a) $\mathbf{1}_S$ (b) Euler-Fourier transform of $\mathbf{1}_S$ (c) $\mathbf{1}_S - \mathbf{1}_C$ (d) Euler-Fourier transform of $\mathbf{1}_S - \mathbf{1}_C$

Figure 1: Euler-Fourier transform of the constructible functions $\mathbf{1}_S$ and $\mathbf{1}_S - \mathbf{1}_C$. The square $S$ is represented by the light blue solid square and the closed curve $C$ is represented by the dark blue dotted curve.

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[1] Not knowing if figures are allowed, I add one without it being necessary for the understanding of the abstract. Please forget this last sentence if figures are not allowed.

Distributed Persistence: Inverse Theorems and Dimensionality Reduction

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What is the "right" topological invariant of a large point cloud $X$? Prior research has focused on estimating the full persistence diagram of $X$, a quantity that is very expensive to compute, unstable to outliers, and far from injective. We therefore propose that, in many cases, the collection of persistence diagrams of many small subsets of X is a better invariant. This invariant, which we call distributed persistence, is embarrassingly parallelizable, more stable to outliers, and has a rich inverse theory. The map from the space of metric spaces (with the quasi-isometry metric) to the space of distributed persistence invariants (with the Hausdorff-Bottleneck distance) is globally bi-Lipschitz. This is a much stronger property than simply being injective, as it implies that the inverse image of a small neighborhood is a small neighborhood, and is to our knowledge the only result of its kind in the TDA literature. By combining distributed persistence with a local, metric term, we introduce a novel approach to dimensionality reduction called DIPOLE. DIPOLE almost surely converges and performs well against popular methods like UMAP, t-SNE, and Isomap on a number of datasets, both visually and in terms of precise quantitative metrics.



Title: Persistent Extension and Analogous Bars: Data-Induced Relations Between Persistence Barcodes

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Abstract:
A central challenge in topological data analysis is the interpretation of barcodes. The classical algebraic-topological approach to interpreting homology classes is to build maps to spaces whose homology carries semantics we understand and then to appeal to functoriality. However, we often lack such maps in real data; instead, we must rely on a cross-dissimilarity measure between our observations of a system and a reference. We will present a pair of computational homological algebra approaches for relating persistent homology classes and barcodes: persistent extension, which enumerates potential relations between cycles from two complexes built on the same vertex set, and the method of analogous bars, which utilizes persistent extension and the witness complex built from a cross-dissimilarity measure to provide relations across systems. Time permitting, we will demonstrate the use of these methods in studying neural population coding and structure propagation on synthetic and real neuroscience datasets.

# The Persistent Topology of Optimal Transport Based Metric Thickenings

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**Abstract**

A metric thickening of a given metric space $X$ is any metric space admitting an isometric embedding of $X$. Thickenings have found use in applications of topology to data analysis, where one may approximate the shape of a dataset via the persistent homology of an increasing sequence of spaces. We introduce two new families of metric thickenings, the $p$-Vietoris–Rips and $p$-Čech metric thickenings for all $1 \leq p \leq \infty$, which include all probability measures on $X$ whose $p$-diameter or $p$-radius is bounded from above, equipped with an optimal transport metric. The $p$-diameter (resp. $p$-radius) of a measure is a certain $\ell_p$ relaxation of the usual notion of diameter (resp. radius) of a subset of a metric space. These families recover the previously studied Vietoris–Rips and Čech metric thickenings when $p = \infty$. As our main contribution, we prove a stability theorem for the persistent homology of $p$-Vietoris–Rips and $p$-Čech metric thickenings, which is novel even in the case $p = \infty$. In the specific case $p = 2$, we prove a Hausmann-type theorem for thickenings of manifolds, and we derive the complete list of homotopy types of the 2-Vietoris–Rips thickenings of the $n$-sphere as the scale increases.



# Beyond field coefficients: saecular barcodes and generators for persistent homology

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January 29, 2022

**Abstract**

A persistence module is a functor $f : \mathbf{I} \to \mathsf{E}$, where $\mathbf{I}$ is the poset category of a totally ordered set. We introduce *saecular decomposition*: a categorically natural method to decompose $f$ into simple parts, called interval modules. Saecular decomposition exists under generic conditions, e.g., when $\mathbf{I}$ is well ordered and $\mathsf{E}$ is a category of modules or groups. This represents a substantial generalization of existing factorizations of 1-parameter persistence modules, leading to, among other things, persistence diagrams not only in homology, but in homotopy.

Applications of saecular decomposition include inverse and extension problems involving filtered topological spaces, the 1-parameter generalized persistence diagram, and the Leray-Serre spectral sequence. Several examples – including cycle representatives for generalized barcodes – hold special significance for scientific applications.

The key tools in this approach are modular and distributive order lattices, combined with Puppe exact categories. An accompanying paper may be found at https://arxiv.org/abs/2112.04927.

# Complexity of computing homology of semi-algebraic sets and mappings

Saugata Basu

**Abstract**

The algorithmic complexity of the problem of computing the homology groups of semi-algebraic sets, and the related problem of computing semi-algebraic triangulations, has been studied for a long time. In this talk I will report on some new progress. The improvement in complexity (for any fixed dimensional homology) measured in terms of the number and degrees of the polynomials appearing in the input formula describing the given semi-algebraic set, as well as the number of variables – goes from doubly exponential, to singly exponential, to even polynomial (in the presence of extra properties like symmetry). If time permits I will describe some applications to computing persistent barcodes of semi-algebraic filtrations, and computing semi-algebraic basis of homology groups of semi-algebraic sets.

Parts of the work are joint (separately) with Negin Karisani, Sarah Percival and Cordian Riener.

# The Many Faces of the Interleaving Distance

Elizabeth Munch

**Abstract**

One might argue that the reason we are all here for this conference and the reason that TDA took off at all is because the topological signatures we study (persistence in all its forms, Reeb graphs, mapper graphs, merge trees, etc) are stable representations of data even in the presence of noise. That is to say, given a ground truth topological space and the noisy, measurable version of that data, the topological representations resulting from the two are at least as similar as the truth and the approximation. For that sentence to make sense at all, we require a metric on the topological signatures. The interleaving distance arose as a natural generalization of the persistence diagram bottleneck distance to its more algebraic counterpart, the persistence module. From there, categorical representations of the same settings have led to a vast field of options for input representation types where the interleaving distance can be applied. And beyond this, many standard $L$ style distances can be realized as an interleaving distance in a properly chosen category. In this talk, we will give a sense of the wide array of available options, with a particular focus on the interleaving distance for graph- based representations of data; and (time-permitting) discuss a new framework for measuring the quality of approximate interleavings.

This work builds on the work of many (Chazal, Cohen-Steiner, Glisse, Guibas, Oudot, Lesnick, Bubenik, Scott, Bjerkevik, Bauer, Robinson, et al) and my collaborations with even more (Percival, B Wang, Chambers, Ophelders, Curry, Botnan, Stefanou, Bollen, Levine, de Silva, Patel, et al).

# TDA and motion planning

Gunnar Carlsson

**Abstract**

Many problems in motion planning can be formulated as (1) identifying the topology of complements and (2) using that information to find paths (perhaps optimal in some sense) connecting two points. This involves understanding unstable homotopy types of complements. One approach to the problem is via the use of the added structure on cohomology around cup products. We will discuss an approach to this which can be implemented computationally.

This represents joint work with Brad Nelson, John Carlsson, Ben Filippenko, and Wyatt Mackey.

# Day 3
# 22.6

## Discrete cubical homotopy groups and real $K(\pi, 1)$ spaces.

Hélène Barcelo

**Abstract**

In this talk we wish to demonstrate how a theory, developed entirely for the purpose of solving problems stemming from search-and-rescue missions, gave rise to one that in turn has applications to fundamental mathematics.

Discrete cubical homotopy theory is a discrete analogue of (singular) simplicial homotopy theory, associating a bigraded sequence of groups to a simplicial complex, capturing some of its combinatorial structure. The motivation for this construction came initially from the desire to find invariants for dynamic processes that were encoded using (combinatorial) simplicial complexes. The invariants should be topological in nature, but should also be sensitive to the combinatorics encoded in the complex, in particular to the level of connectivity among simplices.

Over the last few years similar notions have arisen from several areas of mathematics (e.g., geometric group theory, coarse geometry, computer science) signaling both the pressing need for such a theory as well as its universal nature. As an illustration, we will provide a real analogue of Brieskorn's result on complex $K(\pi, 1)$ spaces: the fundamental group of the complement, over $\mathbb{C}$ of the type $W$ Coxeter arrangement is isomorphic to the pure Artin group of type $W$. In the real case, the fundamental group of the complement, over $\mathbb{R}$, of the $3-$parabolic subspaces arrangement of type $W$ is isomorphic to the discrete cubical homotopy group of the associated simplicial complex.

# Tracking the time evolution of soft matter systems via structural heterogeneity

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Soft matter systems have two common features: high propensity to deformation under small mechanical or thermal stress, and high complexity of their components (liquid crystals, polymers, biological tissues, etc). These characteristics make the mathematical modelling of soft materials challenging and, in most cases, powerful analytic methods are required for an accurate and quantitative characterisation of their dynamics.

In this talk we will discuss a persistent homology framework to track the dynamical behaviour of a wide range of semi-ordered soft matter systems. In particular, we will present an application to the study of phase transitions in nematic liquid crystal nanocomposites. We will show that structural heterogeneity, a topological characteristic for semi-ordered soft materials, can capture their degree of organisation at a mesoscopic level and track their time-evolution, ultimately detecting the order-disorder transition at the microscopic scale. The results presented will show that structural heterogeneity can reveal the effect of the system's geometry on the dynamics of the nematic-isotropic and isotropic-nematic phase transitions, and uncover physical differences between these thermodynamic processes.

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# Section complexes of simplicial height functions

Melvin Vaupel, Erik Hermansen and Paul Trygsland

Consider a real valued continuous function $f\colon X \to \mathbb{R}$. In applications this function could encode a quantity such as time or cost. We are interested in sections of $f$ - that is maps $\sigma\colon [a,b] \to X$ such that $f \circ \sigma = \mathrm{id}$. Homotopical information about sections may be combined across height levels to recover the homotopy type of the base space $X$. This is possible for smooth Morse functions [CJS92], Reeb functions [Try21], but also discrete Morse functions on simplicial complexes [NTT18]. In all of these cases, the homology of $X$ can be computed with a spectral sequence associated to an appropriate topological category of sections. In [VHT22a], we explain how to extract *Reeb complexes* from the first page of such a spectral sequence. These can be understood as generalisations of the Reeb graph to higher homology and encode how generators of homology flow between fibers of $f$ along sections. We prove: if $f$ is a Morse type function, the Reeb complexes reduce to zigzag modules, which relate to the levelset zigzag of $f$ [CdSM09] via the diamond principle [CdS10].

In our paper [VHT22b], that we would like to present at the conference, a *completely combinatorial* and in particular *algorithmically implementable* theory of sections for piecewise linear functions is developed. This is accomplished by modelling such functions as maps of simplicial sets $h\colon X \to \mathrm{R}$, into an appropriate simplicial model of the real line. In analogy to the continuous case, homotopical information about sections of $h$ can be organised across height levels in a bisimplicial set, that we call the *section complex*. We prove that the spectral sequence associated to the section complex always computes the homology of the base space $X$. Furthermore, we can compute the continuous Reeb complexes from above with our simplicial theory of sections. An appropriate proposition, that bridges between the two theories up to homology, is proven.

## References

[CdS10] Gunnar Carlsson and Vin de Silva, *Zigzag persistence*, Foundations of computational mathematics **10** (2010), no. 4, 367–405.

[CdSM09] Gunnar Carlsson, Vin de Silva, and Dmitriy Morozov, *Zigzag persistent homology and real-valued functions*, Proceedings of the twenty-fifth annual symposium on Computational geometry, 2009, pp. 247–256.

[CJS92] Ralph L Cohen, John DS Jones, and Graeme B Segal, *Morse theory and classifying spaces*, preprint (1992).

[NTT18] Vidit Nanda, Dai Tamaki, and Kohei Tanaka, *Discrete Morse theory and classifying spaces*, Advances in Mathematics **340** (2018), 723–790.

[Try21] Paul Trygsland, *Combinatorial models for topological Reeb spaces*, preprint arXiv:2109.05474 [math.AT] (2021).

[VHT22a] Melvin Vaupel, Erik Hermansen, and Paul Trygsland, *Reeb complexes and topological persistence*, preprint (2022).

[VHT22b] ______, *Section complexes of simplicial height functions*, preprint (2022).



## Universal Distribtuion of Persistent Cycles

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One of the fundamental challenges in using persistent homology is how to determine which features are statistically significant, and which are merely noise. A way to address this problem is by characterizing the distribution of the noisy cycles.

In this talk we will consider the empirical distribution of the multiplicative persistence values (i.e. death/birth). We will argue that in random geometric complexes, the persistence distribution of the noise is universal, in the sense that it depends on neither the underlying space nor the original distribution of the point cloud. This statement is currently an open conjecture, but we will present strong experimental evidence for it (in both simulated and real data), as well as heuristic explanations for the source of this phenomenon. We will also demonstrate how this universal distribution can be used to compute p-values for persistent cycles, with very little knowledge about the underlying model.

ract title:

rsistent homology approach for the surveillance of emerging adaptive mutations in the evolution of the corona


ors:

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ract text:

COVID-19 pandemic has initiated an unprecedented worldwide effort to characterize its evolution through the oing of mutations in the genome of the coronavirus SARS-CoV-2. The appearance of new variants of concern, lik nple the Omicron variant, demonstrates that the early identification of mutations that could confer adaptive ntages to the virus, such as higher infectivity or immune evasion, is of paramount importance. However, the lar ber of currently available genomes, several millions at this moment, precludes the efficient use of standard ogeny-based methods. Here we present a new topological approach, and establish a fast and scalable early war m based on persistent homology for the identification and surveillance of emerging adaptive mutations in large mic datasets. Our method systematically detects convergent events in viral evolution merely by their topologic print and thus overcomes limitations of current phylogenetic inference techniques. As a particular mathematica ure, thanks to our use of highly optimized algorithms it easily scales to hundreds of thousands of distinct genom ntroduce a new topological measure for convergent evolution. Analyzing millions of SARS-CoV-2 genomes from ID database, we demonstrate that topologically salient mutations are linked with an increase in infectivity or une escape. As we demonstrate, our method can detect adaptive mutations at an early stage, well before they me recognizable by their prevalence in the population. We report on current emerging potentially adaptive tions, and pinpoint mutations in variants of concern that are likely due to convergent evolution. Our approach ove the surveillance of mutations of concern, guide experimental studies, and aid vaccine development.


rint link:

://arxiv.org/abs/2106.07292

# Sparse Higher Order Čech Filtrations


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January 30, 2022


**Abstract**

The $k$ fold filtration at parameter $r$, whose nerve is the $k$th order Čech filtration at $r$, is formed by the union of intersections of $k$ balls with radius $r$. These balls are usually centered at points of a given point set $P$. We provide an approximation algorithm for the $k$th fold filtration, which in particular leads to a $(1+\epsilon)$-approximation of the higher order Čech complex of $P$. Our techniques are inspired by the approaches for the case $k=1$ of [1, 2, 3, 4]. However, those approaches rely on the removal of vertices according to properties over the vertices themselves, which leads to a size with linear dependency on the number of vertices. For the higher order case, a direct adaptation would result in size bounds of order $n^k$. To overcome this issue, we work with the concept of $k$-distances over $P$. Our approach allows us to define adequate net-like structures over $P$ with cover and packing properties that hold for sets of $k$ points.

At radius $r$, the $k$ fold filtration can be understood as a section of a growing cone-like shape. Each of these cone shapes corresponds to the intersection of a fixed set of $k$ balls. We sparsify the construction by iteratively inactivating points of $P$. During the inactivation of a point $p \in P$, all the cone shapes involving $p$ are first frozen at their state at scale $r$ and later cut off at scale $r' > r$. The order of inactivation and the scales $r$ and $r'$ are given by our net-like structure. The resulting approximate filtration has size linear in $n$ rather than $n^k$. The maximal number of $p$-simplices is

$$\mathcal{O}\left(nk^{k(p+1)}\left(\frac{8(1+\epsilon)^2}{\epsilon}\right)^{2\delta k(p+1)}\right),$$

where $\delta$ is the doubling dimension of the underlying space. We also provide an algorithm for computing the sparsified filtration.


## References


[1] M. Buchet, F. Chazal, S. Y. Oudot, and D. R. Sheehy. Efficient and robust persistent homology for measures. *Comput. Geom.*, 58:70–96, 2016.

[2] N. J. Cavanna, M. Jahanseir, and D. Sheehy. A geometric perspective on sparse filtrations. In *Proceedings of the 27th Canadian Conference on Computational Geometry, CCCG 2015, Kingston, Ontario, Canada, August 10-12, 2015*. Queen's University, Ontario, Canada, 2015.

[3] T. K. Dey, F. Fan, and Y. Wang. Computing topological persistence for simplicial maps. In S. Cheng and O. Devillers, editors, *30th Annual Symposium on Computational Geometry, SoCG'14, Kyoto, Japan, June 08 - 11, 2014*, page 345. ACM, 2014.

[4] D. R. Sheehy. Linear-size approximations to the vietoris-rips filtration. *Discret. Comput. Geom.*, 49(4):778–796, 2013.

# Local Inference of Morse Indices from Finite Point Samples

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Abstract. The Morse index associated to a critical point of a smooth function is a local quantity that is equal to the number of negative eigenvalues of the Hessian evaluated at that point. As second derivatives might be difficult to compute or unavailable in real world contexts, one can use the fundamental results of Morse theory to bypass the need for second derivatives. Unfortunately, this involves computing the relative homology of pairs of sub-level sets, which is no longer a local quantity. In this talk, we propose a new algorithm which combines the best of both worlds by reducing the computation of the Morse index to a homology inference problem. The key ingredient is the theory of Gromoll-Meyer pairs, which facilitates this transition from global sublevel sets to local submanifolds with corners. Finally, we describe an upper bound on the density of sample points needed in order to recover the Morse index.

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**Title:** Quantifying topological features in microscopy images

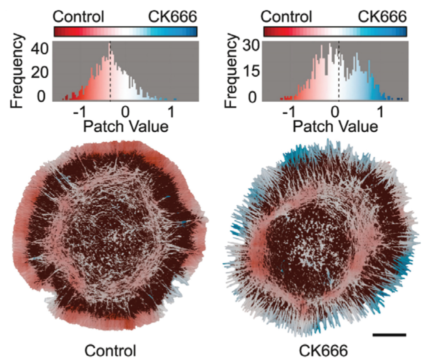
**Abstract:** Microscopy images from biological experiments often depict rich topological and geometric structure. Convolutional neural networks are natural candidates for extracting these features, but their remarkable success has mostly centered on automating tasks that human experts can already perform. Interpretable analyses that aid experts in generating new hypotheses from small experiments remain challenging.

In this talk, I will present new topological data analysis (TDA) image features and a machine-learning-based image analysis pipeline we have introduced, TDAExplore, which performs "weakly supervised" image segmentation of microscopy image data. The pipeline takes grayscale images labeled by experimental group (e.g. control and modified) as input. As output it produces a pixel mask for each image that highlights regions which exhibit topological structures characteristic of each group.

Our TDA image features take advantage of several TDA methods to extract robust information from small image sub-regions: persistent local homology from point clouds, persistence landscapes, and alpha complexes. This combination was engineered for inexpensive computations and the pipeline is highly parallelizable. For example, a typical 8 CPU lab computer running TDAExplore can analyze 70 high resolution images in less than 15 minutes using 5GB of memory.

I will also discuss results from applying our methodology to fluorescence microscopy images of cells' actin cytoskeletons. We investigated the effects of experimentally-induced regulatory changes on actin morphology. The results correctly recapitulate the effects of well-studied regulatory changes and suggest new hypotheses for others. We also obtained whole image classification results that compare favorably with previous studies of benchmark datasets. These datasets feature a variety of microscopy modes and subjects.

Our pipeline demonstrates how modern TDA and machine learning methods can provide accessible and interpretable tools for practitioners in a "small data" setting. TDAExplore is available as R packages with both programmatic and command line interfaces.




Pixel masks of topological scores learned by TDAExplore from actin cytoskeleton data.

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Ordering Topological Descriptor Types

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Given a simplicial complex embedded in $R^d$, there exist finite sets of topological descriptors generated by lower-star filtrations in various directions that, together, faithfully represent the complex. This fact is the foundation of many recent developments in shape representation and comparison. By computing e.g., Euler characteristic curves (ECCs) that arise from filtrations over shapes and some set of standardized directions, the shapes can be compared through the comparison of the ECCs, and even through putting these ECCs in a machine learning pipeline.

Given a simplicial complex and a set of descriptors that faithfully represents it, there is always a minimum cardinality for such a set. Generally, this exact minimum is difficult to know. However, these minimums can be discussed and bounded theoretically, which serves as a measure for how powerful a descriptor type is.

With this motivation, we build a framework through which descriptor types---Euler characteristic curves, persistence diagrams, etc.---can be ordered by their ability to represent shapes. Specifically, we use the size of faithful sets of parameterized descriptors to define this ordering. We then partially order six common descriptor types and discuss the benefits of viewing this work through the lens of constructible cosheaves over a simplicially stratified "sphere of directions." We also discuss a simplicial complex construction for which the minimum set of augmented descriptors needed to form a faithful set is surprisingly large.

# Multivariate Normal Approximations for Simplex Counts in Random Complexes

Tadas Temčinas[*], Vidit Nanda[†], Gesine Reinert[‡]

January 2022

Acyclic partial matchings on simplicial complexes play an important role in topological data analysis by facilitating efficient computation of (persistent) homology groups [5, 8, 6, 7]. In this work we describe probabilistic properties of critical simplex counts for such (lexicographical) matchings on clique complexes of Bernoulli random graphs. This random variable, which arises very naturally in stochastic topology [3, 2], has been poorly studied from a distributional approximation perspective. To the best of our knowledge, only the expected value has been calculated [1, Section 8].

In order to understand the distribution of critical simplex counts, we provide an abstract multivariate central limit theorem using Stein's method [4]. As a consequence of this general result, we are able to extract central limit theorems not only for critical simplex counts, but also for simplex counts in the link of a fixed simplex in a random clique complex. The results quantify the quality of an appropriate normal approximation when the number of vertices in the random complex is finite. Also, the hypotheses of the theorems allow different parameters of the model to depend on each other, giving results in a wide range of parameter combinations.

This talk is mainly based on the recent work [9].

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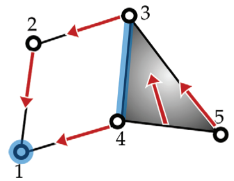


Figure 1: Lexicographical matching given by the red arrows. Critical simplices are highlighted in blue.



## References

[1] Ulrich Bauer and Abhishek Rathod. "Parameterized inapproximability of Morse matching". In: *arXiv:2109.04529* (2021).

[2] Omer Bobrowski and Matthew Kahle. "Topology of random geometric complexes: a survey". In: *Journal of Applied and Computational Topology* 1.3 (2018), pp. 331–364.

[3] Omer Bobrowski and Dmitri Krioukov. "Random Simplicial Complexes: Models and Phenomena". In: *arXiv:2105.12914* (2021).

[4] Louis HY Chen, Larry Goldstein, and Qi-Man Shao. *Normal Approximation by Stein's Method*. Springer, 2011.

[5] Robin Forman. "A user's guide to discrete Morse theory". In: *Séminaire Lotharingien de Combinatoire* 48 (2002), B48c.

[6] Gregory Henselman-Petrusek and Robert Ghrist. "Matroid Filtrations and Computational Persistent Homology". In: *arXiv:1606.00199* (2016).

[7] Leon Lampret. "Chain complex reduction via fast digraph traversal". In: *arXiv:1903.00783* (2019).

[8] Konstantin Mischaikow and Vidit Nanda. "Morse theory for filtrations and efficient computation of persistent homology". In: *Discrete & Computational Geometry* 50.2 (2013), pp. 330–353.

[9] Tadas Temčinas, Vidit Nanda, and Gesine Reinert. "A Multivariate CLT for Dissociated Sums with Applications to U-Statistics and Random Complexes". In: *arXiv preprint arXiv:2112.08922* (2021).

# The Gromov-Hausdorff distance between spheres.

Facundo Mémoli

**Abstract**

The Gromov-Hausdorff distance is a fundamental tool in Riemanian geometry, due the topology it generates, and also in applied geometry and topological data analysis, as a metric for expressing the stability of the persistent homology of geometric data (e.g. via the Vietoris-Rips filtration). Whereas it is often easy to estimate the value of the distance between two given metric spaces, its precise value is rarely easy to determine. Some of the best estimates follow from considerations actually related to both the stability of persistent homology and to Gromov's filling radius. However, these turn out to be non-sharp.

In this talk I will describe these estimates and also results which permit calculating the precise value of the Gromov-Hausdorff between certain pairs of spheres (endowed with their geodesic distance). These results involve lower bounds, which arise from a certain version of the Borsuk-Ulam theorem that is applicable to discontinuous maps, and from matching upper bounds which are induced from specialized constructions of "correspondences" between spheres.

# Persistent homology using filtered closure spaces

Peter Bubenik

**Abstract**

We develop persistent homology in the setting of filtered (Cech) closure spaces. Examples of filtered closure spaces include filtered topological spaces, metric spaces, weighted graphs, and weighted directed graphs. We use various products and intervals for closure spaces to obtain six homotopy theories, six cubical singular homology theories and three simplicial singular homology theories. Applied to filtered closure spaces, these homology theories produce persistence modules. We extend the definition of Gromov-Hausdorff distance to filtered closure spaces and use it to prove that these persistence modules and their persistence diagrams are stable. We also extend the definitions Vietoris-Rips and Cech complexes to give functors on closure spaces and prove that their persistent homology is stable. The Vietoris-Rips functor has a left adjoint which we call the star functor; in contrast the Cech functor does not have a left or right adjoint.

This is joint work with Nikola Milicevic.

# Day 4
# 23.6

# Applying topological data analysis to pure mathematics

Leonard Polterovich

**Abstract**

Topological persistence provides new tools for studying oscillations of functions, e.g., eigenfunctions of the Laplacian, and functionals, e.g., the action functional in classical mechanics. This leads to a number of applications to function theory, spectral geometry, symplectic topology, and dynamical systems.

Based on joint works with Lev Buhovsky, Michael Entov, Jordan Payette, Iosif Polterovich, Egor Shelukhin, and Vukasin Stojisavljevic.

# THE SHIFT-DIMENSION: AN ALGEBRAIC INVARIANT OF MULTIPERSISTENCE MODULES

WOJCIECH CHACHÓLSKI, RENÉ CORBET, AND **ANNA-LAURA SATTELBERGER**

Persistent homology of a one-parameter filtration is algebraically well understood; by a basic structure theorem from algebra, its homology module is uniquely determined by its barcode [6] from which one reads the birth and death times of topological features. The study of *multi*-filtered simplicial complexes and their homology [2] allows to extract finer information from data, but is algebraically intricate. In contrast to the case of a single parameter, there is no discrete complete invariant: as pointed out in [2], the respective moduli space is not zero-dimensional. Moreover, one encounters a lack of stable, algorithmic invariants.

In [3], we investigate an invariant of multipersistence modules that is based on the hierarchical stabilization of discrete invariants of [5]. This construction turns a discrete invariant into a measurable real-valued function in a stable way. The hierarchical stabilization of the zeroth total multigraded Betti number $\beta_0$ is commonly referred to as *stable rank*. In our article, we focus on the stabilization of $\beta_0$ in the direction of a vector. We call the resulting invariant the *shift-dimension* of $M$. We investigate its algebraic properties such as (non)-additivity and the behavior for short exact sequences. The shift-dimension naturally translates to an invariant of multigraded modules over the multivariate polynomial ring.

The computation of the shift-dimension is algorithmic, but in general NP-hard [5]. We give a linear-time algorithm for interval modules in the bivariate case. Direct sums of such modules arise as homology of certain multifiltrations [4] or as approximation of arbitrary finitely presented multipersistence modules in the bivariate case [1].

In summary, we provide a new invariant of persistence modules in the multivariate case which might serve as a feature map for machine learning tasks. It would be my great honor and pleasure to present our work in the section "Multivariate Persistent Homology" at ATMCS10.

## References

[1] H. Asashiba, M. Buchet, E. G. Escolar, K. Nakashima, and M. Yoshiwaki. On interval decomposability of 2D persistence modules. Preprint arXiv:1812.05261, 2018.

[2] G. Carlsson and A. Zomorodian. The theory of multidimensional persistence. *Discrete Comput. Geom.*, 42:71–93, 2009.

[3] W. Chachólski, R. Corbet, and A.-L. Sattelberger. The shift-dimension of multipersistence modules. Preprint arXiv:2112.06509, 2021. Submitted to *SIAM J. Appl. Algebra Geometry.*

[4] E. G. Escolar and Y. Hiraoka. Persistence modules on commutative ladders of finite type. *Discrete Comput. Geom.*, 55(1):100–157, 2016.

[5] O. Gäfvert and W. Chachólski. Stable invariants for multidimensional persistence. Preprint arXiv:1703.03632, 2017.

[6] R. Ghrist. Barcodes: the persistent topology of data. *Bull. Amer. Math. Soc.*, 45:61–75, 2008.

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## Topological Learning from Dynamics on Data

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Topology concerns with those parameters which are preserved under continuous deformation of a space. The number of k-dimensional holes is one main such parameter and computing them has been a challenging question for many years; we are looking for ways to reduce the size of data while not losing its main global (Topological) features: in this regard, the fundamental theories presented by Morse (1925) and Floer (1988) for gradient vector fields and their corresponding discrete versions (originally introduced by Forman, 1998) have become very main tools for qualitative shape analysis for both smooth and discrete settings in the past decade. However a very main challenge in both theory and applications is to generalise these powerful theories and methods to non-gradient vector fields where the (isolated) invariant sets are more complicated than critical points (simplexes). In this seminar, I will talk about how to recover homology groups of both smooth and combinatorial settings, in a unifying perspective, based on dynamical systems operating on the object where we can have periodic orbits as well as critical points. This is the first step to generalise our methods from gradient vector fields to general vector fields ( which also arise in variety of applications) for both smooth and discrete structures.
This is based on a work under the supervision of professor Jost; https://arxiv.org/abs/2105.02567

## HOMOLOGICAL PERCOLATION ON A TORUS

PAUL DUNCAN

Two central objects of study in percolation theory are site percolation and bond percolation, which are the random graphs induced by taking each vertex or edge respectively, independently at random with probability $p$ from the underlying graph. We consider the topology of random cell complexes that generalize each of these models within a large torus $\mathbb{T}^d$. Bond percolation in the integer lattice $\mathbb{Z}^d$, is generalized to plaquette percolation, in which the full $(i-1)$-hypercubical skeleton is included, and then $i$-cells are added randomly. Site percolation on the triangular lattice is equivalent to a random subset of cells in the hexagonal tiling of the plane, which can be generalized to a random subset of the permutohedral tiling of higher dimensional space. Bobrowski and Skraba defined homological percolation within a random subspace as the appearance of a "giant cycle", or a cycle in the random subcomplex that remains nontrivial under the map on homology induced by inclusion into the ambient space. We show that homological percolation in our models exhibits a sharp phase transition for each dimension at which all possible giant cycles appear, with a 1-dimensional threshold that is consistent with classical percolation. Moreover, in the case of percolation in dimension $i = d/2$, the phase transition appears at $p = 1/2$, a higher dimensional analogue to the classical Harris-Kesten theorem. This is joint work with Matthew Kahle at the Ohio State University (mkahle@math.osu.edu) and Benjamin Schweinhart at George Mason University (bschwei@gmu.edu).

# The Generalized Persistence Diagram Encodes the Bigraded Betti Numbers

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February 1, 2022

**Abstract**

We show that the generalized persistence diagram (introduced by Kim and Mémoli) encodes the bigraded Betti numbers of finite 2-parameter persistence modules [1]. More interestingly, we show that the bigraded Betti numbers can be visually read off from the generalized persistence diagram in a manner parallel to how the bigraded Betti numbers are extracted from interval decomposable modules. Our results imply that *all* of the invariants of 2-parameter persistence modules that are computed by the software RIVET are encoded in the generalized persistence diagram. In addition, we verify that a certain recent invariant of finite 2-parameter persistence modules that was introduced by Asashiba et al. also encodes the bigraded Betti numbers.

## References

[1] W. Kim and S. Moore. The generalized persistence diagram encodes the bigraded betti numbers. *arXiv preprint arXiv:2111.02551*, 2021.

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# Extracting Persistent Clusters in Dynamic Data via Möbius inversion

**Woojin Kim**[1] and Facundo Mémoli[2]

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January 30, 2022

**Abstract**

Identifying and representing *clusters* in time-varying network data is of particular importance when studying collective behaviors emerging in nature, in mobile device networks or in social networks. Based on combinatorial, categorical, and persistence theoretic viewpoints, we establish a stable functorial pipeline for the summarization of the evolution of clusters in a time-varying network.

We first construct a complete summary of the evolution of clusters in a given time-varying network over a set of entities $X$ of which takes the form of a *formigram*. This formigram can be understood as a certain Reeb graph $\mathscr{R}$ which is labeled by subsets of $X$. By applying Möbius inversion to the formigram in two different manners, we obtain two dual notions of diagram: the *maximal group diagram* and the *persistence clustergram*, both of which are in the form of an 'annotated' barcode. The maximal group diagram consists of time intervals annotated by their corresponding *maximal groups* — a notion due to Buchin et al., implying that we recognize the notion of maximal groups as a special instance of *generalized persistence diagram* by Patel. On the other hand, the persistence clustergram is mostly obtained by annotating the intervals in the zigzag barcode of the Reeb graph $\mathscr{R}$ with certain merging/disbanding events in the given time-varying network.

We show that both diagrams are complete invariants of formigrams (or equivalently of *trajectory grouping structure* by Buchin et al.) and thus contain more information than the Reeb graph $\mathscr{R}$. This is joint work with Facundo Mémoli. A preprint is available in https://arxiv.org/abs/1712.04064 [v5].

# TOPOLOGY OF RANDOM 2-DIMENSIONAL CUBICAL COMPLEXES

MATTHEW KAHLE, ELLIOT PAQUETTE, AND ÉRIKA ROLDÁN

Abstract. We study a natural model of random 2-dimensional cubical complex which is a subcomplex of an $n$-dimensional cube, and where every possible square 2-face is included independently with probability $p$. Our main result exhibits a sharp threshold $p = 1/2$ for homology vanishing as $n \to \infty$. This is a 2-dimensional analogue of the Burtin and Erdős–Spencer theorems characterizing the connectivity threshold for random graphs on the 1-skeleton of the $n$-dimensional cube.

Our main result can also be seen as a cubical counterpart to the Linial–Meshulam theorem for random 2-dimensional simplicial complexes. However, the models exhibit strikingly different behaviors. We show that if $p > 1 - \sqrt{1/2} \approx 0.2929$, then with high probability the fundamental group is a free group with one generator for every maximal 1-dimensional face. As a corollary, homology vanishing and simple connectivity have the same threshold, even in the strong "hitting time" sense. This is in contrast with the simplicial case, where the thresholds are far apart. The proof depends on an iterative algorithm for contracting cycles — we show that with high probability the algorithm rapidly and dramatically simplifies the fundamental group, converging after only a few steps.

**Extended Abstract: Main Results.** Denote the $n$-dimensional cube by $Q^n = [0,1]^n$, and the set of vertices of the $n$-dimensional cube by $Q_0^n$. This makes $Q_0^n = \{0,1\}^n$, which is the set of all $n$-tuples with binary entries. More generally, denote by $Q_k^n$ the $k$-skeleton of $Q^n$. For example, $Q_1^n$ is the graph with vertex set $Q_0^n$ and an edge (a 1-face) between two vertices if and only if they differ by exactly one coordinate. Define the random 2-dimensional cubical complex $Q_2(n,p)$ as having 1-skeleton $Q_1^n$ and including each 2-dimensional face of $Q^n$ independently with probability $p$.

*Date*: January 31, 2022.

*Key words and phrases.* stochastic topology, cubical complexes, random groups.

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The third author was supported in part by NSF-DMS #1352386 and NSF-DMS #1812028. She has also received funding from the European Union's Horizon 2020 research and innovation program under the Marie Skłodowska-Curie grant agreement No 754462.





The space $Q_2(n,p)$ is a cubical analogue of the random simplicial complex $Y_2(n,p)$ introduced by Linial and Meshulam in [3], whose theory is well–developed. The random complex $Y_2(n,p)$ is defined by taking the complete 1-skeleton of the $n$-dimensional simplex $\Delta^n$, and including into it each 2-face independently and with probability $p$. In this way, $Q_2(n,p)$ is constructed in exactly the same way as $Y_2(n,p)$, *except* that the underlying polytope $\Delta^n$ is replaced by $Q^n$. The space $Q_2(n,p)$ is also a 2-dimensional version of the random cubical graph studied by Burtin [1], Erdős and Spencer [2], and others. More precisely, let $Q(n,p)$ denote the random subgraph defined by including all vertices of $Q^n$, i.e. $Q_0^n$, and including each edge in $Q_1^n$ independently with probability $p$. One can view $Q(n,p)$ as a natural cubical analogue of $G(n,p)$, the *Bernoulli* or *Erdős-Rényi* random graph. In the rest of this abstract we state our main results.

**Theorem 0.1.** *With $Q \sim Q_2(n,p)$ if $p > 1/2$, then $\pi_1(Q) = 0$ asymptotically almost surely. Conversely, if $p \leq 1/2$, then whp there are finitely generated groups $G$ and $F$ so that $\pi_1(Q) \cong G * F$ and where $F$ is a free group of rank at least 2.*

**Theorem 0.2.** *For $p > 1 - (\frac{1}{2})^{1/2}$, with high probability, for $Q \sim Q_2(n,p)$*

$$\pi_1(Q) \cong \underbrace{(\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z})}_{N},$$

*where $N$ denotes the number of isolated 1-faces in $Q$.*

## References

[1] J. D. Burtin. The probability of connectedness of a random subgraph of an $n$-dimensional cube. *Problemy Peredači Informacii*, 13(2):90–95, 1977. ISSN 0555-2923. (Russian).

[2] P. Erdős and J. Spencer. Evolution of the $n$-cube. *Comput. Math. Appl.*, 5(1):33–39, 1979. ISSN 0898-1221. doi: 10.1016/0898-1221(81)90137-1. URL https://doi-org.proxy.lib.ohio-state.edu/10.1016/0898-1221(81)90137-1.

[3] N. Linial and R. Meshulam. Homological connectivity of random 2-complexes. *Combinatorica*, 26(4):475–487, 2006. ISSN 0209-9683. doi: 10.1007/s00493-006-0027-9. URL https://doi-org.proxy.lib.ohio-state.edu/10.1007/s00493-006-0027-9.

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## QUANTIFYING THE HOMOLOGY OF PERIODIC SIMPLICIAL COMPLEXES

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Spatially periodic point patterns are important scientific models, for example of atomic positions in crystalline materials. Associated periodic graphs and simplicial complexes also arise when modelling any large homogeneous data set (c.f., [1, 4]). These infinite complexes are described efficiently using a finite quotient space and translation group action. Both models stem from the familiar covering space relationship between $\mathbb{R}^n$ and the $n$-dimensional torus.

Two eminent goals in studying the finite quotient spaces that arise from periodic simplicial complexes include

- classification of finite representations, and
- extrapolation of topological structure from the quotient.

Classification is challenging because the translation group is not unique; this has been studied extensively for the case of connected graphs (c.f., [3]). Structure extrapolation is complicated by the potential for cycles in a periodic complex to disappear in the quotient space, or conversely for the quotient space to have *toroidal* cycles which do not lift to a cycle in the periodic complex. This can cause weird effects such as disconnected periodic complexes having connected quotient spaces, or contractible periodic complexes having non-contractible quotient spaces.

The ultimate goal is to recover the homology of a periodic simplicial complex from a relatively small quotient space and minimal additional data. We focus on how to identify and distinguish toroidal cycles in a quotient space from *true* cycles of the periodic complex, and how to identify and construct cycles of the periodic complex that become trivial in the quotient space.

In the case of periodic graphs we show that by endowing edges in a quotient graph with appropriate weights in $\mathbb{Z}^d$ as per [2] we can entirely reconstruct the 0- and 1-dimensional homology. For higher dimensional complexes there is no natural analogue of these weights, and instead we introduce a new application of *Mayer-Vietoris spectral sequences* to provide a heuristic to identify toroidal cycles in periodic simplicial complexes.

### References

[1] Catlow, C. R. A. Computational modelling as a tool in structural science.. *IUCrJ.* **7(Pt 5)**, 778 (2020).
[2] Chung, S. J., Hahn, T. & Klee, W. Nomenclature and generation of three-periodic nets: the vector method. *Acta Crystallographica Section A: Foundations of Crystallography.* **40(1)**, 42–50 (1984).
[3] Delgado-Friedrichs, O. & O'Keeffe, M. Identifcation of and symmetry computation for crystal nets. *Acta Crystallographica Section A: Foundations of Crystallography.* **59(4)**, 351–360 (2003).
[4] Iwano, K. & Steiglitz, T. Optimization of one-bit full adders embedded in regular. *IEEE transactions on acoustics, speech, and signal processing.* **34(5)**, 1289–1300 (1986).

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# Signal Compression and Reconstruction on Chain Complexes with Morsified Deformation Retracts

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**Abstract**

At the intersection of Topological Data Analysis and machine learning, the field of cellular signal processing has advanced rapidly in recent years [1, 5]. In this context, a signal is a (co)chain in a (co)chain complex endowed with a degree-wise inner product, and is processed using the combinatorial Laplacian and its associated Hodge decomposition.

The main goal of this paper is to reduce and reconstruct a based chain complex together with a set of signals in such a way that minimizes their reconstruction error. Our approach is rooted in tools of algebraic discrete Morse theory [6], which is able to efficiently generate deformation retracts that reduce the size of the complex while preserving its global topological structure. For this reason, discrete Morse theory has been widely used to speed up computations of (persistent) homology by reducing the size of complexes [4].

In this paper, we explore how such deformation retracts compress and reconstruct signals on the complex. Specifically, we prove that parts of a signal's Hodge decomposition are preserved under compression and reconstruction for specific classes of discrete Morse deformation retracts of a given based chain complex.

Understanding Hodge decomposition of signals requires a careful study of how the choice of algebraic decomposition – or base – of a based chain complex interacts with the reconstruction of the Hodge decomposition components under discrete Morse matchings. As part of this study, we show that any deformation retract of a real finite-dimensional chain complex is equivalent to a Morse matching in some base.

Finally, we provide an algorithm to compute Morse matchings that minimize the reconstruction error for any inner product. We perform several experiments that support our theoretical results and show that our algorithm significantly outperforms randomly generated matchings. We believe that such algorithms could be used in pooling layers of simplicial neural networks [2, 3].

## References


[1] Sergio Barbarossa and Stefania Sardellitti, *Topological signal processing over simplicial complexes*, IEEE Transactions on Signal Processing **68** (2020), 2992–3007.

[2] Cristian Bodnar, Fabrizio Frasca, Yu Guang Wang, Nina Otter, Guido Montúfar, P. Lio', and M. Bronstein, *Weisfeiler and Lehman go topological: Message passing simplicial networks*, Proceedings of the 38th International Conference on Machine Learning PMLR **139**, 1026–1037.

[3] Stefania Ebli, Michaël Defferrard, and Gard Spreemann, *Simplicial neural networks*, Topological Data Analysis and Beyond workshop at NeurIPS, 2020.

[4] Konstantin Mischaikow and Vidit Nanda, *Morse theory for filtrations and efficient computation of persistent homology*, Discrete & Computational Geometry **50** (2013), no. 2, 330–353.

[5] Michael T. Schaub, Yu Zhu, Jean-Baptiste Seby, T. Mitchell Roddenberry, and Santiago Segarra, *Signal processing on higher-order networks: Livin' on the edge... and beyond*, Signal Processing **187** (2021), 108149.

[6] Emil Sköldberg, *Algebraic Morse theory and homological perturbation theory*, Algebra Discrete Math. **26** (2018), 124–129.


# The chromatic number of random Borsuk graphs


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We study a model of random graph where vertices are $n$ i.i.d. uniform random points on the unit sphere $S^d$, and a pair of vertices is connected if the geodesic distance between them is at least $\pi - \varepsilon$. We are interested in the chromatic number of this graph as $n$ tends to infinity.

Our main point is that if $\varepsilon \to 0$ slowly enough as $n \to \infty$, then topological lower bounds on chromatic number are tight. The idea of using topological obstructions to the chromatic number of graphs dates back to Lóvasz in 1978, when he used such constructions to prove Kneser's conjecture. This contrasts with the situation studied by Kahle in 2007, where topological lower bounds are not efficient for the chromatic number of Erdős–Rényi random graphs.

It is not too hard to see that if $\varepsilon > 0$ is small and fixed, then the chromatic number is $d + 2$ with high probability. We show that this holds even if $\varepsilon \to 0$ slowly enough. We quantify the rate at which $\varepsilon$ can tend to zero and still have the same chromatic number. The proof depends on combining topological methods (namely the Lyusternik–Schnirelman–Borsuk theorem) with geometric probability arguments. The rate we obtain is best possible, up to a constant factor — if $\varepsilon \to 0$ faster than this, we show that the graph is $(d + 1)$-colorable with high probability.

Finally, we briefly discuss how this construction can be generalized to other metric spaces where instead of having an antipodality action, we have a free $G$-action, for any finite group $G$. In this setting, rather than the LSB-Theorem, the topological obstructions arise from studing the induced $G$-action on the associated Hom complex.



**Keywords:** random graphs, topological combinatorics, Borsuk–Ulam.

# What are weather regimes? A topologist's answer

Nina Otter

**Abstract**

It has long been suggested that the mid-latitude atmospheric circulation possesses what has come to be known as "weather regimes", which can roughly be categorised as regions of phase space with above-average density. Their existence and behaviour have been extensively studied in meteorology and climate science, due to their potential for drastically simplifying the complex and chaotic mid-latitude dynamics. Several well-known, simple non-linear dynamical systems have been used as toy-models of the atmosphere in order to understand and exemplify such regime behaviour. Nevertheless, no agreed-upon and clear-cut definition of a "regime" exists in the literature, and unambiguously detecting their existence in the atmospheric circulation is often hindered by the high dimensionality of the system.

In this talk I will first give an overview of some of the approaches used to study and define weather regimes. I will then proceed to propose a definition of weather regime that equates the existence of regimes in a dynamical system with the existence of non-trivial topological structure of the system's attractor. I will discuss how this approach is computationally tractable, practically informative, and identifies the relevant regime structure across a range of examples.

This talk is based on joint work with Kristian Strommen, Matthew Chantry and Joshua Dorrington.

# Thoughts on Teaching Topology

Vin de Silva

**Abstract**

I have taught classes in topology for many years now. I will share some thoughts on how I approach it, and how my teaching differs from what I experienced as an undergraduate in the early 1990s, while drawing on what I experienced in primary school in the late 1970s. In particular, I will outline a development of ideas in which homology theory seems to invent itself, and in which cohomology theory is presented simultaneously in an essential way.